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THREE-DIMENSIONAL MOTION OF A VISCOUS INCOMPRESSIBLE FLUID IN A NARROW TUBE

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An approximate solution of the problem of unsteady motion of a viscous incompressible fluid in a long narrow deformable tube at low Reynolds numbers is obtained. Pressure oscillations and tube deformation are shown to be related by an integrodifferential equation. The solution obtained extends the Poiseuille solution in elliptic tubes to the case of comparatively arbitrary small deformations in terms of the tube length and angle.

Key words: *viscous incompressible fluid, Navier–Stokes equations, analytical solution, Poiseuille flow.*

Introduction. Studying flows in blood vessels with various pathological changes, such as aneurism (local swelling of the vessel) and stenosis (local constriction of the vessel), is of particular interest in hemodynamics. There are certain difficulties in studying these processes because of their unsteadiness induced by the pulsed character of blood motion. Axisymmetric analytical solutions [1–3] do not ensure a sufficiently accurate description of real processes. Numerical and experimental modeling [4] is rather expensive (because the flow is three-dimensional and unsteady) and does not always allow parameters affecting the process considered to be defined. Possibly, some questions associated with specific features of blood motion in blood vessels (arteries and arterioles) can be answered with the help of analytical solutions obtained in [5] and in the present paper.

Equations of Motion. Let us consider a three-dimensional unsteady motion of a viscous incompressible fluid. In cylindrical coordinates r, φ, and z, the system of equations has the form [6]

$$\rho\Big(\frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial r}+\frac{v}{r}\,\frac{\partial u}{\partial\varphi}+w\,\frac{\partial u}{\partial z}-\frac{v^2}{r}\Big)=-\frac{\partial p}{\partial r}+\mu\Big(\nabla^2u-\frac{u}{r^2}-\frac{2}{r^2}\,\frac{\partial v}{\partial\varphi}\Big),$$

$$\rho\Big(\frac{\partial v}{\partial t}+u\,\frac{\partial v}{\partial r}+\frac{v}{r}\,\frac{\partial v}{\partial\varphi}+w\,\frac{\partial v}{\partial z}+\frac{uv}{r}\Big)=-\frac{1}{r}\,\frac{\partial p}{\partial\varphi}+\mu\Big(\nabla^2v+\frac{2}{r^2}\,\frac{\partial u}{\partial\varphi}-\frac{v}{r^2}\Big),$$

$$\rho\Big(\frac{\partial w}{\partial t}+u\,\frac{\partial w}{\partial r}+\frac{v}{r}\,\frac{\partial w}{\partial\varphi}+w\,\frac{\partial w}{\partial z}\Big)=-\frac{\partial p}{\partial z}+\mu\nabla^2w, \qquad (1)$$

$$\frac{\partial\,(ru)}{\partial r}+\frac{\partial v}{\partial\varphi}+\frac{\partial\,(rw)}{\partial z}=0,$$

where $\nabla^2=\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\,\dfrac{\partial}{\partial r}+\dfrac{1}{r^2}\,\dfrac{\partial^2}{\partial^2\varphi}+\dfrac{\partial^2}{\partial z^2}$, μ is the dynamic viscosity, $\rho=\text{const}$ is the density, and w, u, and v are the axial, radial, and angular components of the velocity vector, respectively.

The fluid velocity components on the tube wall $[r=r_w(t,\varphi,z)]$ are equal to the wall velocities along the normal, along the tangent, and along the wall:

$$u=\frac{\partial r_w}{\partial t},\qquad v=\frac{\partial^2 r_w}{\partial t\,\partial\varphi},\qquad w=\frac{\partial}{\partial t}\Big(r_w\,\frac{\partial r_w}{\partial z}\Big). \qquad (2)$$

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The following conditions should be satisfied on the tube axis ($r = 0$):

$$u = v = 0. \tag{3}$$

Small Parameters. The following transformations of variables are applied to Eqs. (1):

$$t = \frac{r_0}{U}\,\tilde{t}, \quad r = r_0\tilde{r}, \quad z = \lambda\tilde{z} = \frac{r_0}{\varkappa}\,\tilde{z}, \quad u = U\tilde{u}, \quad v = \frac{U}{\varkappa}\,\tilde{v}, \quad w = \frac{U}{\varkappa}\,\tilde{w}, \quad p = \frac{\mu U}{r_0\varkappa^2}\,\tilde{p} \tag{4}$$

(r_0 is the characteristic size along the radial coordinate, λ is the characteristic size along the longitudinal coordinate, U is the characteristic velocity, and $\varkappa = r_0/\lambda$).

System (1) in the dimensionless variables (4) acquires the form

$$\begin{gathered}
\varkappa^2 \operatorname{Re}\Big(\frac{\partial\tilde{u}}{\partial\tilde{t}} + \tilde{u}\,\frac{\partial\tilde{u}}{\partial\tilde{r}} + \frac{\tilde{v}}{\varkappa\tilde{r}}\,\frac{\partial\tilde{u}}{\partial\varphi} + \tilde{w}\,\frac{\partial\tilde{u}}{\partial\tilde{z}} - \frac{\tilde{v}^2}{\varkappa^2\tilde{r}}\Big) = -\frac{\partial\tilde{p}}{\partial\tilde{r}} + \varkappa^2\Big(\tilde{\nabla}^2\tilde{u} - \frac{\tilde{u}}{\tilde{r}^2} - \frac{2}{\varkappa\tilde{r}^2}\,\frac{\partial\tilde{v}}{\partial\varphi}\Big),\\
\varkappa \operatorname{Re}\Big(\frac{\partial\tilde{v}}{\partial\tilde{t}} + \tilde{u}\,\frac{\partial\tilde{v}}{\partial\tilde{r}} + \frac{\tilde{v}}{\varkappa\tilde{r}}\,\frac{\partial\tilde{v}}{\partial\varphi} + \tilde{w}\,\frac{\partial\tilde{v}}{\partial\tilde{z}} + \frac{\tilde{u}\tilde{v}}{\tilde{r}}\Big) = -\frac{1}{\tilde{r}}\,\frac{\partial\tilde{p}}{\partial\varphi} + \varkappa\tilde{\nabla}^2\tilde{v} + \frac{2\varkappa^2}{\tilde{r}^2}\,\frac{\partial\tilde{u}}{\partial\varphi} - \frac{\varkappa\tilde{v}}{\tilde{r}^2},\\
\varkappa \operatorname{Re}\Big(\frac{\partial\tilde{w}}{\partial\tilde{t}} + \tilde{u}\,\frac{\partial\tilde{w}}{\partial\tilde{r}} + \frac{\tilde{v}}{\varkappa\tilde{r}}\,\frac{\partial\tilde{w}}{\partial\varphi} + \tilde{w}\,\frac{\partial\tilde{w}}{\partial\tilde{z}}\Big) = -\varkappa\,\frac{\partial\tilde{p}}{\partial\tilde{z}} + \varkappa\tilde{\nabla}^2\tilde{w}, \qquad \frac{\partial\,(\tilde{r}\tilde{u})}{\partial\tilde{r}} + \frac{1}{\varkappa}\,\frac{\partial\tilde{v}}{\partial\varphi} + \frac{\partial\,(\tilde{r}\tilde{w})}{\partial\tilde{z}} = 0,
\end{gathered} \tag{5}$$

where $\tilde{\nabla}^2 = \dfrac{\partial^2}{\partial\tilde{r}^2} + \dfrac{1}{\tilde{r}}\,\dfrac{\partial}{\partial\tilde{r}} + \dfrac{1}{\tilde{r}^2}\,\dfrac{\partial^2}{\partial\varphi^2} + \varkappa^2\dfrac{\partial^2}{\partial\tilde{z}^2}$ and $\operatorname{Re} = \rho U r_0/\mu$ is the Reynolds number.

Let $\operatorname{Re} \to 0$ and $\varkappa^2 \to 0$. After passing to the limit as $\operatorname{Re} \to 0$ and $\varkappa^2 \to 0$, Eqs. (5) reduce to the following system of equations:

$$\begin{gathered}
\frac{\partial\tilde{p}}{\partial\tilde{r}} = -\frac{2\varkappa}{\tilde{r}^2}\,\frac{\partial\tilde{v}}{\partial\varphi}, \qquad \frac{1}{\tilde{r}}\,\frac{\partial\tilde{p}}{\partial\varphi} = \varkappa\Big(\frac{\partial^2\tilde{v}}{\partial\tilde{r}^2} + \frac{1}{\tilde{r}}\,\frac{\partial\tilde{v}}{\partial\tilde{r}} + \frac{1}{\tilde{r}^2}\,\frac{\partial^2\tilde{v}}{\partial\varphi^2} - \frac{\tilde{v}}{\tilde{r}^2}\Big),\\
\frac{\partial\tilde{p}}{\partial\tilde{z}} = \frac{\partial^2\tilde{w}}{\partial\tilde{r}^2} + \frac{1}{\tilde{r}}\,\frac{\partial\tilde{w}}{\partial\tilde{r}} + \frac{1}{\tilde{r}^2}\,\frac{\partial^2\tilde{w}}{\partial\varphi^2}, \qquad \varkappa\,\frac{\partial\,(\tilde{r}\tilde{u})}{\partial\tilde{r}} + \frac{\partial\tilde{v}}{\partial\varphi} + \varkappa\frac{\partial\,(\tilde{r}\tilde{w})}{\partial\tilde{z}} = 0.
\end{gathered} \tag{6}$$

Let us also assume that

$$\frac{\partial\tilde{p}}{\partial\tilde{r}} = 0, \qquad \frac{\partial\tilde{v}}{\partial\varphi} \sim \varkappa. \tag{7}$$

Then, we can cancel the terms $\varkappa\,\partial\tilde{v}/\partial\varphi$ and $\varkappa\,\partial^2\tilde{v}/\partial\varphi^2$ in the first and second equations, respectively, and retain the term $\partial\tilde{v}/\partial\varphi$ in the fourth equation of system (6). In dimensional variables, system (6) is written in the form

$$\begin{gathered}
\frac{\partial p}{\partial r} = 0, \qquad \frac{1}{\mu r}\,\frac{\partial p}{\partial\varphi} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\,\frac{\partial v}{\partial r} - \frac{v}{r^2},\\
\frac{1}{\mu}\,\frac{\partial p}{\partial z} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\,\frac{\partial w}{\partial r} + \frac{1}{\tilde{r}^2}\,\frac{\partial^2 w}{\partial\varphi^2}, \qquad \frac{\partial\,(ru)}{\partial r} + \frac{\partial v}{\partial\varphi} + \frac{\partial\,(rw)}{\partial z} = 0.
\end{gathered} \tag{8}$$

Solution of the Problem. System (8) has a general solution in the form

$$\begin{gathered}
u(t,r,\varphi,z) = -\frac{r}{4\mu}\Big[\ln\Big(\frac{r}{r_0}\Big) - 1\Big]\frac{\partial^2 p}{\partial\varphi^2} - \frac{r}{2}\,\frac{\partial C(t,\varphi,z)}{\partial\varphi},\\
v(t,r,\varphi,z) = \frac{r}{4\mu}\Big[2\ln\Big(\frac{r}{r_0}\Big) - 1\Big]\frac{\partial p}{\partial\varphi} + rC(t,\varphi,z),\\
w(t,r,\varphi,z) = -\frac{1}{4\mu}\,\frac{\partial p}{\partial z}\,(r_0^2 - r^2) + \varkappa\,\frac{p_1(t)}{4\mu r_0}\,r^2\Phi(\varphi),
\end{gathered} \tag{9}$$

where $\Phi(\varphi) = A\cos(2\varphi) - B\sin(2\varphi)$, A and B are arbitrary constants, and $p_1(t)$ is a function of time. The function $C(t,\varphi,z)$ and the derivative of pressure $\partial p(t,\varphi,z)/\partial\varphi$ are found from the boundary conditions (2) as follows:

$$C(t,\varphi,z) = \frac{1}{r_w}\,\frac{\partial^2 r_w}{\partial t\,\partial\varphi} - \frac{1}{4\mu}\Big[2\ln\Big(\frac{r_w}{r_0}\Big) - 1\Big]\frac{\partial p(t,\varphi,z)}{\partial\varphi},$$

$$\frac{\partial p(t,\varphi,z)}{\partial\varphi} = \frac{4\mu}{r_w^2}\int\Big(2r_w\,\frac{\partial r_w}{\partial t} - \frac{\partial r_w}{\partial\varphi}\,\frac{\partial^2 r_w}{\partial t\,\partial\varphi} + r_w\,\frac{\partial^3 r_w}{\partial t\,\partial\varphi^2}\Big)\,d\varphi. \tag{10}$$

Solution (9) satisfies system (8) under the following conditions for pressure:

$$\frac{\partial^3 p}{\partial z\,\partial\varphi^2} = 0, \qquad \frac{\partial^2 p}{\partial z^2} = 0. \tag{11}$$

Conditions (11) are equivalent to the pressure function of the form

$$p(t,\varphi,z) = p_1(t)z/\lambda + p_2(t,\varphi) + p_3(t). \tag{12}$$

In this case, the function of wall deformation is

$$r_w(t,\varphi,z) = r_0F(z)G(t,\varphi). \tag{13}$$

Thus, the pressure $p_2(t,\varphi)$ and the function $G(t,\varphi)$ of wall deformation $r_w(t,\varphi,z)$ (13) are related by

$$\frac{\partial p_2(t,\varphi)}{\partial\varphi} = \frac{4\mu}{G^2}\int\Big(2G\,\frac{\partial G}{\partial t} - \frac{\partial G}{\partial\varphi}\,\frac{\partial^2 G}{\partial t\,\partial\varphi} + G\,\frac{\partial^3 G}{\partial t\,\partial\varphi^2}\Big)\,d\varphi. \tag{14}$$

By virtue of the boundary condition (2) for the longitudinal velocity w, the product of the functions $F(z)G(t,\varphi)$ should satisfy the equation

$$\frac{\partial F^2G^2}{\partial t\,\partial z} = \frac{\varkappa p_1(t)}{2\mu r_0}\,\{F^2G^2[\Phi(\varphi)+1] - 1\}. \tag{15}$$

Equation (15) has the solution

$$F(z) = 1 + \varkappa f(z), \qquad G(t,\varphi) = [1 + \varkappa g(t,\varphi)]/\sqrt{\Phi(\varphi)+1}, \tag{16}$$

where $f(z)$ and $g(t,\varphi)$ are arbitrary functions. Solution (16) satisfies Eq. (15) with accuracy to $O(\varkappa^2)$.

General Approximate Solution. With accuracy to terms of the order $O(\varkappa^2)$, the problem solution has the form

$$r_w(t,\varphi,z) = r_0\{1 + \varkappa[f(z) + g(t,\varphi)]\}\Big/\sqrt{\tilde\Phi(\varphi)}; \tag{17}$$

$$p(t,\varphi,z) = p_1(t)z\varkappa/r_0 + p_2(t,\varphi) + p_3(t); \tag{18}$$

$$u(t,r,\varphi) = -\frac{r}{4\mu}\Big[\ln\Big(\frac{r}{r_0}\Big) - 1\Big]\frac{\partial^2 p_2}{\partial\varphi^2} - \frac{r}{2}\,\frac{\partial C(t,\varphi)}{\partial\varphi},$$

$$v(t,r,\varphi) = \frac{r}{4\mu}\Big[2\ln\Big(\frac{r}{r_0}\Big) - 1\Big]\frac{\partial p_2}{\partial\varphi} + rC(t,\varphi), \qquad w(t,r,\varphi) = -p_1(t)[r_0^2 - r^2\tilde\Phi(\varphi)]\varkappa/(4\mu r_0), \tag{19}$$

where

$$\tilde\Phi(\varphi) = 1 + A\cos(2\varphi) - B\sin(2\varphi), \tag{20}$$

A and B are arbitrary constants, $f(z)$, $g(t,\varphi)$, $p_1(t)$, and $p_3(t)$ are arbitrary functions, and the function $p_2(t,\varphi)$ is related to wall deformation by Eq. (10), which acquires the form

$$p_2(t,\varphi) = 4\mu\int\Big[\frac{1}{G^2}\int\Big(2G\,\frac{\partial G}{\partial t} - \frac{\partial G}{\partial\varphi}\,\frac{\partial^2 G}{\partial t\,\partial\varphi} + G\,\frac{\partial^3 G}{\partial t\,\partial\varphi^2}\Big)d\varphi\Big]\,d\varphi,$$

$$G(t,\varphi) = [1 + \varkappa g(t,\varphi)]\Big/\sqrt{\tilde\Phi(\varphi)}; \tag{21}$$

$$C(t,\varphi) = \frac{1}{G(t,\varphi)}\,\frac{\partial^2 G(t,\varphi)}{\partial t\,\partial\varphi} - \frac{1}{4\mu}\Big[2\ln\Big(\frac{r_w}{r_0}\Big) - 1\Big]\,\frac{\partial p_2(t,\varphi)}{\partial\varphi}. \tag{22}$$

One can easily verify that solution (19) for an arbitrary angular velocity $\partial v/\partial\varphi$ satisfies assumption (7).

Thus, we found the general solution (9), (12)–(14), (16) of system (8) with the boundary conditions (2) and conditions on the axis (3), which describes unsteady three-dimensional motion of a viscous incompressible fluid in a deformable tube. This solution describes the flow in a tube at low Reynolds numbers and a small (with accuracy to the second order of smallness) ratio of the transverse and longitudinal characteristic sizes.

Generalized Poiseuille Flow. Let $g(t, \varphi) \equiv g(\varphi)$. Then, we obtain

$$r_w(\varphi, z) = r_0\{1 + \varkappa[f(z) + g(\varphi)]\} \Big/ \sqrt{\tilde{\Phi}(\varphi)}; \tag{23}$$

$$p(t, z) = p_1(t) z\varkappa / r_0 + p_3(t); \tag{24}$$

$$u = 0, \qquad v = 0, \qquad w(t, r, \varphi) = -p_1(t)[r_0^2 - r^2\tilde{\Phi}(\varphi)]\varkappa/(4\mu r_0), \tag{25}$$

where

$$\tilde{\Phi}(\varphi) = 1 + A\cos(2\varphi) - B\sin(2\varphi), \tag{26}$$

A and B are arbitrary constants, and $p_1(t)$ and $p_3(t)$ are arbitrary functions.

At $g(\varphi) \equiv 0$ and $f(z) \equiv 0$, solution (23)–(26) reduces to the known Poiseuille solution of the problem about the flow through a cylindrical tube with an elliptic cross section [1, p. 381]. Indeed, the equation of wall deformation (23) in Cartesian coordinates is the equation of an ellipse

$$(1 + A)x^2 - 2Bxy + (1 - A)y^2 = r_0^2 \tag{27}$$

($x = r\cos\varphi$ and $y = r\sin\varphi$), which is not reduced to the principal axes.

The difference between the generalized Poiseuille solution (23)–(26) and the known solution [6] is as follows: at small values of the parameter $\varkappa$, the generalized solution admits vessel deformation along the z axis [at $f(z) \neq 0$] and over the angle φ [at $g(\varphi) \neq 0$], in addition to the elliptic deformation of the vessel (27).

Peristaltic Flow. A peristaltic flow is a flow in a tube with axially symmetric deformation [1, 2], i.e., $r_w = r_w(t, z)$, and the angular velocity is $v \equiv 0$.

Solution (17)–(22) does not describe a peristaltic flow, because the pressure is a linear function of the longitudinal coordinate. Therefore, the longitudinal velocity w is also independent of the coordinate z, and the fourth equation of system (8) has only the trivial solution $u = 0$. It follows from the conditions $u = v = 0$ for this solution (17)–(22) that $g(t, \varphi) = \text{const}$, and wall deformation is independent of time. Therefore, this solution cannot describe peristaltic motion with $r_w = r_w(t, z)$.

Conclusion. A flow of a viscous incompressible fluid in a deformable tube is considered in this paper. A solution of unsteady three-dimensional Navier-Stokes equations is obtained for a low-Reynolds-number flow in a long narrow tube (under the condition of small deformations of the walls). The solution depends on unsteady deformation of the tube wall and on pressure oscillations. As a particular case, the solution extends the Poiseuille solution in elliptic tubes with a fairly arbitrary small deformation of the wall in terms of length and angle.

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